Math 250 3.5 Derivatives of Trig Functions

Objectives

1) Find derivatives using the quotient rule and trig identities to find the remaining trig derivatives
   a. $\frac{d}{dx}\tan x$
   b. $\frac{d}{dx}\cot x$
   c. $\frac{d}{dx}\sec x$
   d. $\frac{d}{dx}\csc x$
   e. *****Memorize the derivatives of all six trigonometric functions.*****

Recall: Definition of Derivative
$m_{TAN}(x) = f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Examples and Practice:

1) Find the limits analytically
   a. $\lim\limits_{x \to 0} \frac{3 - 3\cos(x)}{x}$
   b. $\lim\limits_{\theta \to 0} \frac{\cos(\theta)\tan(\theta)}{\theta}$
   c. $\lim\limits_{x \to 0} \frac{\tan^2(x)}{x}$
   d. $\lim\limits_{x \to 0} \frac{\sin(6x)}{\sin(8x)}$

2) Use the quotient rule and the derivatives of $f(x) = \sin x$ and $f(x) = \cos x$ to find the derivatives of
   a. $\tan x$
   b. $\cot x$
   c. $\sec x$
   d. $\csc x$

3) Find first derivatives
   a. $y = \frac{1 - \sin x}{\cos x}$
   b. $f(\theta) = (\theta + 1)\cot\theta$
   c. $f(x) = \frac{1}{1 + \csc(x)}$
   d. $f(x) = \frac{x + \sec(x)}{x^2 \tan x}$

4) Find the second derivative of $f(x) = \sec(x)$
5) Find the third derivative of $f(x) = \tan x$

① a) $\lim_{x \to 0} \frac{3(1-\cos x)}{x}$

Try substituting $= \frac{3(1-\cos 0)}{0}$

$= \frac{3(1-1)}{0}$

$= \frac{0}{0}$ indeterminate. ☹ Back to work.

Recall $\lim_{x \to 0} \frac{1-\cos x}{x} = 0$.

Use properties of limits:

$= 3 \cdot \left[\lim_{x \to 0} \frac{1-\cos x}{x}\right]$

$= 3 \cdot 0$ substitute known limit.

$= \boxed{0}$

b) $\lim_{\theta \to 0} \frac{\cos\theta \tan\theta}{\theta}$

$= \lim_{\theta \to 0} \frac{\cancel{\cos}\,\theta}{\theta} \cdot \frac{\sin\theta}{\cancel{\cos\theta}}$ rewrite $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$= \lim_{\theta \to 0} \frac{\sin\theta}{\theta}$ cancel common factor

$= \boxed{1}$ substitute known limit.

c) $\lim_{x \to 0} \frac{\tan^2 x}{x}$

$= \lim_{x \to 0} \frac{1}{x} \cdot \frac{\sin^2 x}{\cos^2 x}$ subst $\tan x = \frac{\sin x}{\cos x}$

$= \left[\lim_{x \to 0} \frac{\sin x}{x}\right] \cdot \left[\lim_{x \to 0} \frac{\sin x}{\cos x}\right] \cdot \left[\lim_{x \to 0} \frac{1}{\cos x}\right]$ split into known limits

$= 1 \cdot 0 \cdot 1$

$= \boxed{0}$

M250

d) $\lim_{x \to 0} \frac{\sin 6x}{\sin 8x}$

$= \lim_{x \to 0} \frac{\sin 6x}{6x} \cdot \frac{6x}{8x} \cdot \frac{8x}{\sin 8x}$

Reduce! $\frac{6x}{8x} = \frac{3}{4}$

$= \frac{\left[\lim_{x \to 0} \frac{\sin 6x}{6x}\right] \cdot \left[\lim_{x \to 0} \frac{3}{4}\right]}{\left[\lim_{x \to 0} \frac{\sin 8x}{8x}\right]}$

$\frac{\sin 6x}{6x} = \frac{\sin y}{y}$ if $y = 6x$
(and $y \to 0$ when $6x \to 0$)
$\lim_{x \to 0} \frac{\sin 6x}{6x} = \lim_{y \to 0} \frac{\sin y}{y} = 1$

$= \frac{1 \cdot \frac{3}{4}}{1}$ ← limit of a constant

substitute known limit twice

$= \boxed{\frac{3}{4}}$

Math 250

Find derivatives

(2) a) $\frac{d}{dx}[\tan x] = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right]$

$$= \frac{\cos x \cdot \frac{d}{dx}[\sin x] - \sin x \cdot \frac{d}{dx}[\cos x]}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$ Pythagorean trig identity.

$$= \frac{1}{\cos^2 x}$$

$$= \boxed{\sec^2 x}$$

b)

$\frac{d}{dx}[\cot x] = \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right]$

$$= \frac{\sin x \cdot \frac{d}{dx}[\cos x] - \cos x \cdot \frac{d}{dx}[\sin x]}{\sin^2 x}$$

$$= \sin x \cdot (-\sin x) - \cos x \cdot \cos x$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$ factor -1

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$ Pythagorean trig identity

$$= \frac{-1}{\sin^2 x}$$

$$= \boxed{-\csc^2 x}$$

Math 250

Find derivatives

c)

$$\frac{d}{dx}[\sec x] = \frac{d}{dx}\left[\frac{1}{\cos x}\right]$$

$$= \frac{\cos x \cdot \frac{d}{dx}[1] - 1 \cdot \frac{d}{dx}[\cos x]}{\cos^2 x}$$

$$= \frac{[\cos x] \cdot 0 - 1 \cdot (-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$$

$$= \boxed{\sec x \cdot \tan x}$$

d)

$$\frac{d}{dx}[\csc x] = \frac{d}{dx}\left[\frac{1}{\sin x}\right]$$

$$= \frac{\sin x \cdot \frac{d}{dx}[1] - 1 \cdot \frac{d}{dx}[\sin x]}{\sin^2 x}$$

$$= \frac{[\sin x] \cdot 0 - 1 \cdot \cos x}{\sin^2 x}$$

$$= \frac{-\cos x}{\sin^2 x}$$

$$= -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}$$

$$= \boxed{-\csc x \cdot \cot x}$$

Math 250

* MEMORIZE TRIG DERIVATIVES *

$$\frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \cot x = -\csc^2 x$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\frac{d}{dx} \csc x = -\csc x \cot x$$

Math 250

yes (3) a) $y = \dfrac{1-\sin x}{\cos x}$

Find y' using quotient rule.
Find y' by rewriting y using $\tan x$ and $\sec x$.
Show that the results from a) and b) are equal.

option 1

$$y' = \frac{[\cos x]\cdot\frac{d}{dx}[1-\sin x] - [1-\sin x]\cdot\frac{d}{dx}[\cos x]}{\cos^2 x}$$ quotient rule

$$= \frac{\cos x\cdot(-\cos x) - (1-\sin x)(-\sin x)}{\cos^2 x}$$ derivatives

$$= \frac{-\cos^2 x + \sin x - \sin^2 x}{\cos^2 x}$$ simplify

$$= \frac{-(\cos^2 x + \sin^2 x) + \sin x}{\cos^2 x}$$ collect trig identity $\sin^2 x + \cos^2 x = 1$

$$= \boxed{\frac{-1+\sin x}{\cos^2 x}}$$

option 2

$$y = \frac{1}{\cos x} - \frac{\sin x}{\cos x}$$ Separate fractions.

$$y = \sec x - \tan x$$ trig identities

$$y' = \sec x\tan x - \sec^2 x$$ differentiate

$$y' = \boxed{\sec x\,(\tan x - \sec x)}$$ factor

option 3

$$\sec x\,(\tan x - \sec x)$$

$$= \frac{1}{\cos x}\left[\frac{\sin x}{\cos x} - \frac{1}{\cos x}\right]$$ trig identities

$$= \frac{1}{\cos x}\left[\frac{\sin x - 1}{\cos x}\right]$$ add fractions w/ CD.

$$= \frac{\sin x - 1}{\cos^2 x} \checkmark$$ mult denoms.

Math 250

(3) b) $f(\theta) = (\theta+1)\cot\theta$

product rule

$$f'(\theta) = (\theta+1)\frac{d}{d\theta}\cot\theta + \frac{d}{d\theta}(\theta+1)\cdot\cot\theta$$

$$= (\theta+1)\cdot(-\csc^2\theta) + 1\cdot\cot\theta$$

$$= \boxed{-\theta\csc^2\theta - \csc^2\theta + \cot\theta}$$

Hint: Do in x if θ is intimidating. But don't forget to write final answer using θ.

$f(x) = (x+1)\cdot\cot x$

product rule

$$f'(x) = (x+1)\cdot\frac{d}{dx}\cot x + \frac{d}{dx}(x+1)\cdot\cot x$$

$$= (x+1)(-\csc^2 x) + 1\cdot\cot x$$

$$= -x\csc^2 x - \csc^2 x + \cot x$$

$\Rightarrow$

$$f'(\theta) = -\theta\csc^2\theta - \csc^2\theta + \cot\theta$$

(3) c) $f(x) = \dfrac{1}{1+\csc x}$

Quotient Rule

$$f'(x) = \frac{(1+\csc x)\cdot\frac{d}{dx}(1) - 1\cdot\frac{d}{dx}(1+\csc x)}{(1+\csc x)^2}$$

$$= \frac{(1+\csc x)\cdot 0 - (0 - \csc x\cot x)}{(1+\csc x)^2}$$

$$= \boxed{\frac{\csc x\cot x}{(1+\csc x)^2}}$$

M250

(3) d) $f(x) = \dfrac{x + \sec x}{x^2 \tan x}$ ← product inside quotient

Quotient structure:

$$f'(x) = \frac{(\text{bottom})(\frac{d}{dx}\text{top}) - (\text{top})(\frac{d}{dx}\overset{\text{product}}{\text{bottom}})}{(\text{bottom})^2}$$

$$f'(x) = \frac{(x^2 \tan x)(1 + \sec x \tan x) - (x + \sec x)\left[(\text{1st})(\frac{d}{dx}\text{2nd}) + (\text{2nd})(\frac{d}{dx}\text{1st})\right]}{(x^2 \tan x)^2}$$

$$= \frac{x^2 \tan x + x^2 \sec x \tan^2 x - (x + \sec x)\left[x^2(\sec^2 x) + \tan x\,(2x)\right]}{x^4 \tan^2 x}$$

$$= \frac{x^2 \tan x + x^2 \sec x \tan x - \left[x^3 \sec^2 x + 2x^2 \tan x + x^2 \sec^3 x + 2x \sec x \tan x\right]}{x^4 \tan^2 x}$$

$$= \frac{x^2 \tan x + x^2 \sec x \tan x - x^3 \sec^2 x - 2x^2 \tan x - x^2 \sec^3 x - 2x \sec x \tan x}{x^4 \tan^2 x}$$

$$= \frac{-x^2 \tan x + x^2 \sec x \tan x - x^3 \sec^2 x - x^2 \sec^3 x - 2x \sec x \tan x}{x^4 \tan^2 x}$$

$$= \frac{\cancel{x}\left[-x \tan x + x \sec x \tan x - x^2 \sec^2 x - x \sec^3 x - 2 \sec x \tan x\right]}{\cancel{x} \cdot x^3 \tan^2 x}$$

$$= \boxed{\frac{-x \tan x + x \sec x \tan x - x^2 \sec^2 x - x \sec^3 x - 2 \sec x \tan x}{x^3 \tan^2 x}}$$

M250

(4) $f(x) = \sec x$

Find $f''(x)$.

find $f'(x) = \sec x \tan x$ (memorized)

$$f''(x) = \sec x \cdot \frac{d}{dx}[\tan x] + \tan x \cdot \frac{d}{dx}[\sec x]$$

$$= \sec x \cdot \sec^2 x + \tan x \cdot \sec x \tan x$$

$$= \sec x \left[\sec^2 x + \tan^2 x\right]$$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\tan^2 x = \sec^2 x - 1$$

$$= \sec x \left[\sec^2 x + \sec^2 x - 1\right]$$

$$= \boxed{\sec x \left[2\sec^2 x - 1\right]}$$

Math 250

YES (5) Find the third derivative of $f(x) = \tan x$.

$f'(x) = \sec^2 x$ (memorized)

$f'(x) = \sec x \cdot \sec x$

[Note: we can also use the chain rule, but that's 2.6]

$f''(x) = \sec x \cdot \frac{d}{dx}[\sec x] + \frac{d}{dx}[\sec x] \cdot \sec x$

$\quad 1\text{st} \cdot \frac{d}{dx}[2\text{nd}] + \frac{d}{dx}[1\text{st}] \cdot 2\text{nd}.$

$f''(x) = \sec x \cdot [\sec x \tan x] + [\sec x \tan x] \sec x$

$= \sec^2 x \tan x + \sec^2 \tan x$

$f''(x) = 2\sec^2 x \tan x.$

$f''(x) = 2 \underbrace{\sec x}_{1\text{st}} \cdot \underbrace{\sec x}_{2\text{nd}} \cdot \underbrace{\tan x}_{3\text{rd}}$

Use product rule for three functions:

$f'''(x) = 2\Big[\frac{d}{dx}[\sec x] \cdot \sec x \cdot \tan x$

$\quad + \sec x \cdot \frac{d}{dx}[\sec x] \cdot \tan x$

$\quad + \sec x \cdot \sec x \cdot \frac{d}{dx}[\tan x] \Big]$

$= 2\Big[\sec x \tan x \cdot \sec x \tan x$

$\quad + \sec x \cdot \sec x \tan x \cdot \tan x$

$\quad + \sec x \cdot \sec x \cdot \sec^2 x \Big]$

$= 2[\sec^2 x \tan^2 x + \sec^2 x \tan^2 x + \sec^4 x]$

$= 2[2\sec^2 x \tan^2 x + \sec^4 x]$

$\boxed{f'''(x) = 2\sec^2 x [2\tan^2 x + \sec^2 x]}$

Identities

$1 + \tan^2 x = \sec^2 x$

$\tan^2 x = \sec^2 x - 1$

OR

$= 2\sec^2 x [2(\sec^2 x - 1) + \sec^2 x]$

$= 2\sec^2 x [2\sec^2 x - 2 + \sec^2 x]$

$\boxed{f'''(x) = 2\sec^2 x [3\sec^2 x - 2]}$

Sometimes identities make it a lot better. Here, not so much.